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Archangelskii's Solution of Alexandrov's Problem

by

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Introduction

The purpose of this paper is to present the proof of a conjecture of P.S. Alexandrov, namely that every first countable compact T_2 space has at most $2^{\aleph_0}$ points. This conjecture is nearly fifty years old and only very recently has it been solved by A.V. Archangelskii (see [1]).

Although the proof we give here is slightly more general and somewhat simpler than Archangelskii's, all the main ideas - or rather tricks - that we use belong to Archangelskii. Thus this paper can be regarded as a quick translation of [1] for those whose knowledge of Russian is not sufficient to read the original.

1.1. Definition. The transfinite sequence $\{p_\xi : \xi < \mu\}$ of points of a space X is called a free sequence if for each $\xi_0 < \mu$

$$\overline{\{p_\xi : \xi < \xi_0\}} \cap \overline{\{p_\xi : \xi_0 \leq \xi < \mu\}} = \emptyset.$$

1.2. Main Lemma. Suppose X is an arbitrary topological space, α is an infinite cardinal and $|X| > 2^\alpha$, and that moreover the following two conditions hold:

(i) If $A \subset X$, $|A| \leq \alpha$ then $|\bar{A}| \leq 2^\alpha$.

(ii) If $A \subset X$, $|A| \leq \alpha$ then $X \setminus \bar{A}$ can be written as a union of at most 2^α closed subsets of X (or what amounts for the same $\psi(\bar{A},X) \leq 2^\alpha$, where $\psi(H,X)$ denotes the minimal cardinality of a system of open sets in X, whose intersection is H).

Then X contains a free sequence of length α^+ (i.e. the successor cardinal of α).

Proof. We shall construct a ramification system in the sense of [2], Lemma 1, by defining sets $R[\rho_0,\ldots,\rho_\xi]$ and points $p[\rho_0,\ldots,\rho_\xi]$ for certain sequences of ordinals where $\rho_\eta < 2^\alpha$ and $\xi < \alpha^+$.

First we put $R_0 = X$ and $p_0 \in R_0$ arbitrary; here 0 stands for the empty sequence. Suppose now that $\xi < \alpha^+$ and for all $\eta < \xi$ the sets $R[\rho_0,\ldots,\rho_\eta]$ and points $p[\rho_0,\ldots,\rho_\eta]$ have been defined for each $[\rho_0,\ldots,\rho_\eta] \in S_{\eta+1}$, where S_ν denotes the set of sequences of type ν of ordinals $< 2^\alpha$.

Let us choose now a sequence $s \in S_\xi$ and put

$$R'_s = \bigcap \{R_{s|\eta+1} : \eta+1 \leq \xi\}$$

where $s|\eta+1$ denotes the initial segment of s of type $\eta+1$. Now we distinguish two cases, a) and b):

a) $|R'_s| \leq 2^\alpha$. In this case we put $R_{[s,\rho]} = R'_s$ for all $\rho < 2^\alpha$; here $[s,\rho]$ denotes the sequence $[\rho_0,\ldots,\rho]$ of type $\xi+1$ obtained by augmenting s by ρ. The points $p_{[s,\rho]}$ can be chosen arbitrarily.

b) $|R'_s| > 2^\alpha$. Since $\xi < \alpha^+$, applying (ii) and putting $\{p_{s|\eta+1} : \eta+1 \le \xi\} = G^{(s)}$ we can write $X \setminus G^{(s)} = \bigcup\{F_\rho^{(s)} : \rho < 2^\alpha\}$, where the $F_\rho^{(s)}$'s are (not necessarily distinct) closed subsets of X. Next we put

$$R_{[s,\rho]} = R'_s \cap F_\rho^{(s)}$$

for each $\rho < 2^\alpha$ and choose any element of $R_{[s,\rho]}$ as $p_{[s,\rho]}$ if $R_{[s,\rho]} \neq \emptyset$. Otherwise $p_{[s,\rho]}$ can be chosen arbitrarily.

By transfinite induction on ν we can easily show that

$$X = \bigcup\{R'_s : s \in S_\nu\} \cup \bigcup\{G^{(s)} : s \in S_\nu\}$$

holds for each $\nu < \alpha^+$. Next we claim that there exists a sequence $t \in S_{\alpha^+}$ such that

$$|R'_{t|\nu}| > 2^\alpha$$

holds for each $\nu < \alpha^+$. Indeed, let us put

$$\tilde{S}_\nu = \{s \in S_\nu : |R'_s| \le 2^\alpha\}$$

and

$$S = \bigcup\{S_\nu : \nu < \alpha^+\}, \quad \tilde{S} = \bigcup\{\tilde{S}_\nu : \nu < \alpha^+\}.$$

Then $|\tilde{S}| \le |S| \le \sum_{\nu<\alpha^+} 2^{|\nu|} \le \alpha^+ . 2^\alpha = 2^\alpha$, hence we have, by (i) and the choice of $\tilde{S}$

$$|\bigcup\{G^{(s)} : s \in S\} \cup \bigcup\{R'_s : s \in \tilde{S}\}| \le 2^\alpha . 2^\alpha + 2^\alpha . 2^\alpha = 2^\alpha.$$

Now if x_0 is an arbitrary point in the complement of the above set we can find a sequence $t \in S_{\alpha^+}$ such that

$$x_0 \in R'_{t|\nu}$$

holds for each $\nu < \alpha^+$. Indeed, if t is a maximal sequence such that $x_0 \in R'_{t|\nu}$ holds for each $\nu <$ length of t, then the length of t must be α^+. Because of the choice of x_0, however, we have $t|\nu \in S_\nu \setminus \tilde{S}_\nu$, hence $|R'_{t|\nu}| > 2^\alpha$ for each $\nu < \alpha^+$.

Let us put now $t = [\rho_0, \dots, \rho_\xi, \dots]$ and

$$p_\xi = p_{t|\xi+1} = p_{[\rho_0,\dots,\rho_\xi]}$$

for all $\xi < \alpha^+$. Then for arbitrary $\xi < \alpha^+$ we have

$$\overline{\{p_\eta : \eta < \xi\}} = G^{(t|\xi)} \quad \text{and}$$

$$\{p_\eta : \xi \leq \eta < \alpha^+\} \subset \overline{\{p_\eta : \xi \leq \eta < \alpha^+\}} \subset F_{\rho_\xi}^{(t|\xi)} ,$$

which shows that $\{p_\xi : \xi < \alpha^+\}$ is a free sequence, because $G^{(t|\xi)} \cap F_{\rho_\xi}^{(t|\xi)} = \emptyset$, by definition. This completes the proof.

2.1. Definition. A space X is called α-Lindelöf if from each open covering of X we can select a subcovering of power $\leq \alpha$.

2.2. Lemma. Assume X is an α-Lindelöf T_1 space, $A \subset X$, $|A| \leq 2^\alpha$ and A is closed in X, moreover that $\psi(p,X) \leq 2^\alpha$ holds for each $p \in A$. Then

$$\psi(A,X) \leq 2^\alpha$$

holds as well.

Proof. Let us choose for each $p \in A$ a system of open neighbourhoods of p, say $\mathcal{V}_p$, such that $\cap \mathcal{V}_p = \{p\}$ and $|\mathcal{V}_p| \leq 2^\alpha$. Now, if x_0 is an arbitrary point of $X \setminus A$ then for each $p \in A$ there is a $V_p \in \mathcal{V}_p$ such that $x_0 \notin V_p$. Since $\{V_p : p \in A\}$ is a covering of A and X (and A) are α-Lindelöf, there is a subcovering $\mathcal{U}_{x_0} \subset \{V_p : p \in A\}$ such that $|\mathcal{U}_{x_0}| \leq \alpha$. But $x_0 \notin \cup \mathcal{U}_{x_0} \supset A$, which shows that

$$\psi(A,X) \leq |\{\mathcal{U} : \mathcal{U} \subset \bigcup_{p \in A} \mathcal{V}_p \text{ and } |\mathcal{U}| \leq \alpha\}| \leq (2^\alpha)^\alpha = 2^\alpha,$$

since $|\bigcup_{p \in A} \mathcal{V}_p| \leq 2^\alpha . 2^\alpha = 2^\alpha$.

2.3. Lemma. Suppose X is a T_2 space and $\chi(X) = \sup\{\chi(p,X) : p \in X\} \leq \alpha$. (Here, as usual, $\chi(p,X)$ denotes the minimal cardinality of a neighbourhood basis of p in X.) Then $A \subset X$, $|A| \leq \alpha$ imply $|\bar{A}| \leq 2^\alpha$.

Proof. Let $p \in \bar{A}$, then there is a Moore-Smith sequence converging to p on an index set of power $\leq \alpha$ whose terms are elements of A. Since X is T_2, for different points these sequences must also be different. Since the number of all such Moore-Smith sequences in A is $\leq 2^\alpha$, we have $|\bar{A}| \leq 2^\alpha$.

2.4. Definition. We define $\mathcal{L}(X)$ as the smallest infinite cardinal such that X is α-Lindelöf.

2.5. Theorem. For each T_2 space X we have

$$|X| \leq 2^{\mathcal{L}(X) . \chi(X)} .$$

Proof. Let us put $\alpha = \mathcal{L}(X).\chi(X)$. By 2.3. and 2.2., respectively, the conditions (i) and (ii) of 1.2. are satisfied. So, by 1.2., if $|X| > 2^{\alpha}$ held, there would exist a free sequence $S = \{p_\xi : \xi < \alpha^+\}$ in X. Since X is α-Lindelöf there is a point $y \in X$ such that for every neighbourhood V of y $|V \cap S| = |S| = \alpha^+$ holds. Indeed this is true in every α-Lindelöf space for every set of power α^+.

On the other hand, since $\chi(y,X) \leq \alpha$ obviously there is a subset $A \subset S$, $|A| \leq \alpha$ such that $y \in \bar{A}$. Now, since α^+ is regular there is a $\xi_0 < \alpha^+$ such that

$$A \subset \{p_\xi : \xi < \xi_0\} ,$$

hence $$y \in \overline{\{p_\xi : \xi < \xi_0\}} .$$

Since S is free we have $y \notin \overline{\{p_\xi : \xi_0 \leq \xi < \alpha^+\}}$ which is in contradiction to the choice of y. This completes the proof.

2.6. Corollary. If X is a first countable Lindelöf T_2 space then $|X| \leq 2^{\aleph_0}$.

References

[1] A.V. Archangelskii, On the cardinality of first countable compacta (In Russian), Dokl. Akad. Naask. SSSR, 187 (1969), No. 5, 967-968.

[2] P. Erdös, A. Hajnal and R. Rado, Partition relations for cardinal numbers, Acta Math. Acad. Sci. Hung., 16 (1965), 93-196.